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Ground states for the higher-order dispersion managed NLS equation in the absence of average dispersion

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Abstract

The problem of existence of ground states in higher-order dispersion managed NLS equation is considered. The ground states are stationary solutions to dispersive equations with nonlocal nonlinearity which arise as averaging approximations in the context of strong dispersion management in optical communications. The main result of this note states that the averaged equation possesses ground state solutions in the practically and conceptually important case of the vanishing residual dispersions.

Keywords: Nonlinear Schrödinger equation; Dispersion management; Ground states

1. Introduction

Over the past 10 years, certain nonlinear dispersive equations with nonlocal nonlinearity have arisen in the context of optical communications and have become the subject of intense numerical and analytical study [5,1,11,21,8,9,12]. In general, these

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equations are of the form

$$u_t = -i\nabla\mathcal{H}(u), \tag{1}$$

where

$$\mathcal{H}(u) = \frac{\alpha}{2}\int_{\mathbb{R}} |u_x|^2 - \frac{1}{4}\int_0^1 \int_{\mathbb{R}} |T(t)u|^4 \, dx\, dt, \tag{2}$$

∇ denotes the Frechét derivative of the Hamiltonian $\mathcal{H}$, and T denotes the solution operator for the linear dispersive equation

$$iu_t = \sum_{m=2}^{M} \beta_m(t)(-i\partial_x)^m u, \tag{3}$$

where the coefficients $\beta_m(t)$ are piecewise constant and periodic with zero mean.

Such equations arise naturally as averaging approximations to the nonlinear dispersive equations that model pulse propagation in dispersion managed (DM) optical fibers [5,1,11], and a question of great interest has been the existence and stability of solitary wave solutions. The first work in this direction was done for the case $M = 2$, which in optical communications is known as conventional dispersion management. It was shown that when $\alpha > 0$, the Hamiltonian $\mathcal{H}$ possesses a ground state in $H^1 = H^1(\mathbb{R}; \mathbb{C})$ [21,8]. A natural extension of this work was to study the variational problem with $\alpha = 0$. This problem, while interesting from an analytical point of view, is also important physically, as certain physical effects that are destabilizing to pulse propagation in an optical fiber are minimized in the regime $\alpha \approx 0$ [18,20]. Due to Strichartz-type estimates for solutions of linear dispersive equations [7], the corresponding Hamiltonian is bounded from below in $L^2 = L^2(\mathbb{R}; \mathbb{C})$. However, loss of compactness of a minimizing sequence could have become a problem, due to potential loss of control on derivatives. Nevertheless, this variational problem was analyzed successfully in [9], where it was shown that vanishing and splitting of the minimizing sequence (in the language of concentration compactness [10]) is not possible in both Fourier and 'physical' space. Hence the problem is essentially localized in Fourier and in physical space (up to L^2-errors which are controlled), and therefore one is back to the classical situation where Sobolev's embedding theorem can be applied. As a result, the minimizing sequence converged to a ground state, strongly in L^2.

Recent advances in manufacture techniques have made it possible to extend dispersion management to higher-order dispersion, and for such a system the appropriate averaged equation is again of the form in (1)–(3), but with $M = 3$. Analysis of the type in [21] was carried out for the case $\alpha > 0$, yielding ground states in $H^1 = H^1(\mathbb{R}; \mathbb{C})$ [12]. Two natural questions come to mind when considering this case. First, can one extend the analysis for $\alpha = 0$ to this equation, and second, is it possible to further extend the analysis to cases of arbitrarily high-order dispersion management ($M > 3$)?

In this paper, we will show that the answer to these questions is affirmative, using the method in [9]. We will also use a technical simplification of the method from [9], relying on a certain multilinear estimate, which was suggested by an anonymous referee of that paper. We will discuss compensation of both even and odd orders without lower order terms, and furthermore mixed cases up to order three. The linear part of the equation has the general form (3). To simplify the exposition, we will assume that all β_m are periodic step-functions, more precisely that $\beta_m(t-1) = \beta_m(t+1)$, $\beta_m(t) = -b_m \neq 0$ for $t \in (-1, 0)$, and $\beta_m(t) = b_m$ for $t \in (0, 1)$ hold. Considering the more general case with β_m being general piecewise constant mean-zero periodic functions does not create any new difficulties, but makes the derivations more cumbersome. In this (symmetric) case and with zero average dispersion, $\alpha = 0$, the Hamiltonian functional of the averaged equation reduces to

$$\mathcal{H}(u) = -\frac{1}{2}\int_0^1 \int_{\mathbb{R}} |T(t)u|^4 \, dx \, dt, \tag{4}$$

where we have used that the integral over the period $(-1, 1)$ is equal to the double value of the integral over $(0, 1)$. In (4) we denoted by $T(t)$ the solution operator of the general equation

$$iu_t = \sum_{m=2}^{M} b_m(-i\partial_x)^m u, \tag{5}$$

which is the above linear equation (3) for $t \in (0, 1)$, and therefore with constant coefficients. Furthermore, $T_m(t)$ stands for the solution operator of the linear equation with the single dispersion term ∂_x^m, i.e., $u(t, x) = (T_m(t)u_0)(x)$ solves

$$iu_t = (-i\partial_x)^m u \tag{6}$$

with initial data $u(0, x) = u_0(x)$.

Our first main result concerns the pure higher-order dispersion case.

Theorem 1.1. *Let $m \geqslant 3$ and $T_m(t)$ be defined via (6). Then the minimization problem*

$$P_{1,m} = \inf\left\{\varphi_m(u) : u \in L^2, \int_{\mathbb{R}} |u|^2 \, dx = 1\right\} < 0, \tag{7}$$

with the functional φ_m given by

$$\varphi_m(u) = -\int_0^1 \int_{\mathbb{R}} |(T_m(t)u)(x)|^4 \, dx \, dt, \quad u \in L^2, \tag{8}$$

possesses a solution $u \in L^2$.

Note that the functional $\mathcal{H}$ from (4) has been renamed to φ_m to allow for an easier comparison with [9], which our strategy of proof follows; we will also use the simplification mentioned above. The main new technical problem compared to [9] results from the fact that in the case $m \geqslant 3$ the functional φ_m is no longer invariant under rotations, i.e., in general $\varphi_m(e^{iax}u) \neq \varphi_m(u)$ for $a \in \mathbb{R}$. Stated differently, φ_m is not invariant under translations of the Fourier transform. The latter property was important in [9], since it allowed us to re-center those minimal sequences which are localized in Fourier space, but whose 'centers' move off to infinity. Due to the lack of invariance of the functional φ_m a new argument had to be found. It turned out, however, that the loss of invariance was beneficial for the construction of a minimizing sequence, as the sequences whose 'centers' move to infinity could be shown to be not minimizing, see Lemma 2.5 below.

We prove the theorem in Section 2 by taking any minimizing sequence and constructing a strongly converging subsequence (up to translation of the original sequence). The first step is to show, in Section 2.1, that there is a subsequence which is tight in the Fourier domain. Then we will verify in Section 2.2 that there is yet another subsequence which (up to translation) is also tight in physical space, from which the strong convergence in L^2 follows.

For the mixed cases up to third order we could obtain a similar result, which in particular yields the existence of a ground state in the motivating problem that was described above.

Theorem 1.2. *Let $T(t)$ denote the solution operator of the equation*

$$iu_t = -b_2\,\partial_x^2 u + ib_3\,\partial_x^3 u, \quad \textit{where} \quad b_2, b_3 \neq 0.$$

Then the minimization problem

$$P_1 = \inf\left\{\varphi(u) : u \in L^2,\ \int_{\mathbb{R}} |u|^2\,dx = 1\right\} < 0,$$

with the functional φ given by

$$\varphi(u) = -\int_0^1 \int_{\mathbb{R}} |(T(t)u)(x)|^4\,dx\,dt, \quad u \in L^2, \tag{9}$$

has a solution $u \in L^2$.

Remark 1.3. Note, that the case $b_3 = 0, b_2 \neq 0$ has been treated in [9] and the case $b_3 \neq 0, b_2 = 0$ follows from Theorem 1.1.

Up to some technical differences, the proof of Theorem 1.2 naturally is quite similar to the proof of Theorem 1.1; it will be carried out in Section 3.

2. Proof of Theorem 1.1

2.1. Tightness of minimizing sequences in the Fourier domain

In this section we establish the tightness of every minimizing sequence in Fourier space, up to selection of a subsequence; see (23) in Corollary 2.8 below for the notion of tightness we are using.

From (6) we obtain the representation

$$(T_m(t)u)(x) = \int_{\mathbb{R}} e^{i(x\xi - t\xi^m)}\hat{u}(\xi)\,d\xi, \tag{10}$$

where here and henceforth for simplicity all 2π-factors in the Fourier transforms are dropped, so that we have $\hat{u}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}u(x)\,dx$. A basic related Strichartz-type estimate is

$$\|T_m(\cdot)u\|_{L^{2(m+1)}_{tx}(\mathbb{R}\times\mathbb{R})} \leqslant C\|u\|_{L^2}, \quad u \in L^2, \tag{11}$$

see [7] or [19, 5.19(b), p. 369] with $n = 1$, $\phi(\xi) = -\xi^m$, $k = m$, $q = 2(m+2)$, and $\alpha = 0$. The following lemma states a certain refined multilinear estimate related to T_m. The usefulness of such type of estimates was explained to the first author by an anonymous referee of [9], who also outlined its application (see Lemmas 2.3 and 2.4 below); this help is gratefully acknowledged. In spirit, Lemma 2.1 is similar to e.g. [16] or [3, Lemma 2.2], where refinements of Strichartz' estimates are discussed. We remark that we did not try to optimize the decay power $q(m)$ in (12); for our purposes it is sufficient to obtain some $q(m) > 0$.

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$\|(T_m(\cdot)u)(T_m(\cdot)v)\|_{L^2_{tx}([0,1]\times\mathbb{R})} \leqslant C\,\mathrm{dist}(I,J)^{-q(m)}\,\|u\|_{L^2}\|v\|_{L^2} \tag{12}$$

for all functions $u, v \in L^2$ such that $\hat{u}$ and $\hat{v}$ are supported in disjoint intervals $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$, respectively, which are at positive distance. For $m \geqslant 2$ the function $q(m) > 0$ is defined by

$$q(m) = \begin{cases} \frac{m-1}{2} & : \ m \text{ is even}, \\ \frac{1}{6} & : \ m \text{ is odd}. \end{cases} \tag{13}$$

Proof. Without loss of generality we may assume that I lies to the left of J. Denoting $a = \sup I$ and $b = \inf J$ thus $\mathrm{dist}(I,J) = b - a > 0$. Writing $u(t) = T_m(t)u$ and

$v(t) = T_m(t)v$ we have from Parseval's identity

$$\|(T_m(\cdot)u)(T_m(\cdot)v)\|^2_{L^2_{tx}([0,1]\times\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} |u(t,x)|^2|v(t,x)|^2 \mathbf{1}_{[0,1]}(t)\,dx\,dt$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(\tau,\xi)G(\tau,\xi)\,d\xi\,d\tau,$$

with

$$\Phi = \mathcal{F}(uv), \quad G = \overline{\mathcal{F}}(\bar{u}\bar{v}\mathbf{1}_{[0,1]}(t))$$

and $\mathcal{F}$ denoting the space-time Fourier transform. In view of (10) thus

$$\Phi(\tau,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(\tau t+\xi x)}u(t,x)v(t,x)\,dx\,dt$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{u}(\xi_1)\hat{v}(\xi_2)\delta_0(\tau+\xi_1^m+\xi_2^m)\delta_0(\xi-\xi_1-\xi_2)\,d\xi_1\,d\xi_2.$$

Consequently, the representation

$$\|(T_m(\cdot)u)(T_m(\cdot)v)\|^2_{L^2_{tx}([0,1]\times\mathbb{R})}$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{u}(\xi_1)\hat{v}(\xi_2)G(-\xi_1^m-\xi_2^m,\xi_1+\xi_2)\,d\xi_1\,d\xi_2 \tag{14}$$

is obtained. Now we consider separately the two different cases.

Case 1: m is even. Here we can use a well-known argument which relies on the gain which is obtained by introducing a suitable transformation. For this we let $\eta = (\eta_1,\eta_2) = (-\xi_1^m - \xi_2^m, \xi_1+\xi_2)$, $d\eta_1 d\eta_2 = m|\xi_2^{m-1} - \xi_1^{m-1}|d\xi_1 d\xi_2$, to get from (14) and Hölder's inequality

$$\|(T_m(\cdot)u)(T_m(\cdot)v)\|^2_{L^2_{tx}([0,1]\times\mathbb{R})}$$
$$\leqslant C\int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{u}(\xi_1(\eta))|\,|\hat{v}(\xi_2(\eta))|\,|G(\eta_1,\eta_2)|\,\frac{d\eta_1 d\eta_2}{|\xi_2(\eta)^{m-1}-\xi_1(\eta)^{m-1}|}$$
$$\leqslant C\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{u}(\xi_1(\eta))|^2\,|\hat{v}(\xi_2(\eta))|^2\,\frac{d\eta_1 d\eta_2}{|\xi_2(\eta)^{m-1}-\xi_1(\eta)^{m-1}|^2}\right)^{1/2}\|G\|_{L^2_{\tau\xi}}$$
$$\leqslant C\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{u}(\xi_1)|^2\,|\hat{v}(\xi_2)|^2\,\frac{d\xi_1 d\xi_2}{|{\xi_2}^{m-1}-{\xi_1}^{m-1}|}\right)^{1/2}\|G\|_{L^2_{\tau\xi}}$$
$$\leqslant C(b^{m-1}-a^{m-1})^{-1/2}\|u\|_{L^2}\|v\|_{L^2}\,\|G\|_{L^2_{\tau\xi}}.$$

Since $m-1$ is odd, $b^{m-1}-a^{m-1}\geqslant C(b-a)^{m-1}=C\,\mathrm{dist}(I,J)^{m-1}$, cf. Lemma 2.2(i) below. Observing

$$\|G\|_{L^2_{\tau\xi}}=\|uv\mathbf{1}_{[0,1]}(t)\|_{L^2_{tx}}=\|(T_m(\cdot)u)(T_m(\cdot)v)\|_{L^2_{tx}([0,1]\times\mathbb{R})},$$

we thus obtain (12) for even m.

Case 2: m is odd, $m=2n+1$. First we are going to argue that without loss of generality we can assume that $b-a\geqslant 1$. Indeed, Hölder's inequality, the elementary inequality $|z|^4\leqslant\varepsilon^{-2}|z|^2+\varepsilon^{2(m-1)}|z|^{2(m+1)}$ with $\varepsilon=\|u\|_{L^2}^{-1}$, and (11) yield

$$\begin{aligned}\|(T_m(\cdot)u)(T_m(\cdot)v)\|_{L^2_{tx}([0,1]\times\mathbb{R})} &\leqslant \|T_m(\cdot)u\|_{L^4_{tx}([0,1]\times\mathbb{R})}\,\|T_m(\cdot)v\|_{L^4_{tx}([0,1]\times\mathbb{R})}\\ &\leqslant C\left(\varepsilon^{-2}\int_0^1\|u\|_{L^2}^2\,dt+\varepsilon^{2(m-1)}\|u\|_{L^2}^{2(m+1)}\right)^{1/4}\\ &\quad\times\left(\varepsilon^{-2}\int_0^1\|v\|_{L^2}^2\,dt+\varepsilon^{2(m-1)}\|v\|_{L^2}^{2(m+1)}\right)^{1/4}\\ &\leqslant C\|u\|_{L^2}\|v\|_{L^2}\,;\end{aligned}$$

observe that $(\widehat{T_m(t)u})(\xi)=e^{-it\xi^m}\hat{u}(\xi)$, whence $T_m(t)$ preserves all H^s-norms. Thus if $b-a\leqslant 1$, then we can produce any factor $1\leqslant(b-a)^{-q}=\mathrm{dist}(I,J)^{-q}$ on the right-hand side. Therefore, we will suppose in the sequel that $b-a\geqslant 1$. Inserting the factor $|\xi_2^{m-1}-\xi_1^{m-1}|^{-1/3+1/3}$ in (14), Hölder's inequality leads to

$$\begin{aligned}&\|(T_m(\cdot)u)(T_m(\cdot)v)\|^2_{L^2_{tx}([0,1]\times\mathbb{R})}\\ &\leqslant C\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\hat{u}(\xi_1)|^{3/2}|\hat{v}(\xi_2)|^{3/2}}{|\xi_2^{m-1}-\xi_1^{m-1}|^{1/2}}\,d\xi_1\,d\xi_2\right)^{2/3}\\ &\quad\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\xi_2^{m-1}-\xi_1^{m-1}||G(-\xi_1^m-\xi_2^m,\xi_1+\xi_2)|^3\,d\xi_1\,d\xi_2\right)^{1/3}\\ &\leqslant C\left(\int_I\int_J\frac{|\hat{u}(\xi_1)|^{3/2}|\hat{v}(\xi_2)|^{3/2}}{|\xi_2^{2n}-\xi_1^{2n}|^{1/2}}\,d\xi_1\,d\xi_2\right)^{2/3}\\ &\quad\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|G(\eta_1,\eta_2)|^3\,d\eta_1\,d\eta_2\right)^{1/3},\end{aligned}$$

where in the last step we have again used the transformation $(\eta_1, \eta_2) = (-\xi_1^m - \xi_2^m, \xi_1 + \xi_2)$. To bound the first term, we note that for $\xi_2 \in J$ and $\xi_1 \in I$ the estimate

$$\begin{aligned} \xi_2^{2n} - \xi_1^{2n} &= (\xi_2^2 - \xi_1^2)\Big((\xi_1^2)^{n-1} + (\xi_1^2)^{n-2}\xi_2^2 + \cdots + \xi_1^2(\xi_2^2)^{n-2} + (\xi_2^2)^{n-1}\Big) \\ &\geqslant |\xi_2 - \xi_1||\xi_2 + \xi_1|\Big(\xi_1^{2(n-1)} + \xi_2^{2(n-1)}\Big) \geqslant C(b-a)|\xi_2 + \xi_1| \end{aligned}$$

follows from $b - a \geqslant 1$, see Lemma 2.2(ii). Therefore the Hardy–Littlewood–Sobolev inequality [17, p. 31] implies

$$\begin{aligned} &\|(T_m(\cdot)u)(T_m(\cdot)v)\|^2_{L^2_{tx}([0,1]\times\mathbb{R})} \\ &\leqslant C\,\mathrm{dist}(I,J)^{-1/3}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\hat{u}(\xi_1)|^{3/2}|\hat{v}(\xi_2)|^{3/2}}{|\xi_2+\xi_1|^{1/2}}\,d\xi_1 d\xi_2\right)^{2/3} \|G\|_{L^3_{\tau\xi}} \\ &\leqslant C\,\mathrm{dist}(I,J)^{-1/3}\left\||\hat{u}|^{3/2}\right\|^{2/3}_{L^{4/3}}\left\||\hat{v}|^{3/2}\right\|^{2/3}_{L^{4/3}}\|G\|_{L^3_{\tau\xi}} \\ &\leqslant C\,\mathrm{dist}(I,J)^{-1/3}\|u\|_{L^2}\|v\|_{L^2}\|G\|_{L^3_{\tau\xi}}. \end{aligned} \tag{15}$$

Thus it remains to estimate $\|G\|_{L^3_{\tau\xi}}$. For this purpose, we note that

$$\begin{aligned} \|G\|_{L^3_{\tau\xi}} &= \|\overline{\mathcal{F}}(\bar{u}\bar{v}\mathbf{1}_{[0,1]}(t))\|_{L^3_{\tau\xi}} \leqslant C\|\bar{u}\bar{v}\mathbf{1}_{[0,1]}(t)\|_{L^{3/2}_{tx}} \\ &= C\left(\int_0^1\int_{\mathbb{R}} |u(t,x)|^{3/2}|v(t,x)|^{3/2}\,dx\,dt\right)^{2/3} \\ &\leqslant C\left(\int_0^1\int_{\mathbb{R}} |u(t,x)|^3\,dx\,dt\right)^{1/3}\left(\int_0^1\int_{\mathbb{R}} |v(t,x)|^3\,dx\,dt\right)^{1/3}. \end{aligned} \tag{16}$$

Using the elementary inequality $|z|^3 \leqslant \varepsilon^{-1}|z|^2 + \varepsilon^{2m-1}|z|^{2(m+1)}$ with $\varepsilon = \|u\|_{L^2}^{-1}$ and (11), we get similarly as before

$$\int_0^1\int_{\mathbb{R}} |u(t,x)|^3\,dx\,dt \leqslant C\left(\varepsilon^{-1}\int_0^1 \|u\|^2_{L^2}\,dt + \varepsilon^{2m-1}\|u\|^{2(m+1)}_{L^2}\right) \leqslant C\|u\|^3_{L^2}.$$

Thus (12) follows from (15) and (16). This completes the proof of Lemma 2.1. □

The following technical lemma has been needed in the above proof.

Lemma 2.2. (i) *Let $n \in \mathbb{N}$ be odd. Then $b^n - a^n \geqslant 2^{1-n}(b-a)^n$ for every $a, b \in \mathbb{R}$ with $b \geqslant a$.* (ii) *Let $k \in \mathbb{N}$. There exists a constant $C > 0$ such that whenever $a, b \in \mathbb{R}$ with $b - a \geqslant 1$, then $\xi_1 \leqslant a$ and $\xi_2 \geqslant b$ implies $\xi_1^{2k} + \xi_2^{2k} \geqslant C$.*

Proof. (i) We have $b^n - a^n = n\int_a^b x^{n-1}\,dx$. If $b \geqslant a \geqslant 0$, then $\int_a^b x^{n-1}\,dx \geqslant \int_0^{b-a} x^{n-1}\,dx = n^{-1}(b-a)^n$. If $b \geqslant 0 \geqslant a$ and $b + a \geqslant 0$, then $\int_a^b x^{n-1}\,dx = \int_a^{(b-a)/2} x^{n-1}\,dx + \int_{(b-a)/2}^b x^{n-1}\,dx \geqslant \int_a^{(b-a)/2} x^{n-1}\,dx + \int_{-a}^{(b-a)/2} x^{n-1}\,dx = \int_{-(b-a)/2}^{(b-a)/2} x^{n-1}\,dx = n^{-1} 2^{1-n}(b-a)^n$. If $b \geqslant 0 \geqslant a$ and $b + a \leqslant 0$, then $\int_a^b x^{n-1}\,dx = \int_{a-(b-a)/2x}^{n-1} dx + \int_{-(b-a)/2}^{b} x^{n-1}\,dx \geqslant \int_{-(b-a)/2}^{-b} x^{n-1}\,dx + \int_{-(b-a)/2}^{b} x^{n-1}\,dx = \int_{-(b-a)/2}^{(b-a)/2} x^{n-1}\,dx = n^{-1}2^{1-n}(b-a)^n$. The last case $a \leqslant b \leqslant 0$ is symmetric to $b \geqslant a \geqslant 0$. (ii) If $b \geqslant a \geqslant 0$, then $b \geqslant 1 + a \geqslant 1$, whence $\xi_1^{2k} + \xi_2^{2k} \geqslant b^{2k} \geqslant 1$. If $b \geqslant 0 \geqslant a$ and $a \leqslant -1/2$, then $\xi_1^{2k} + \xi_2^{2k} \geqslant a^{2k} \geqslant 2^{-2k}$. If $b \geqslant 0 \geqslant a$ and $a \geqslant -1/2$, then $b \geqslant 1 + a \geqslant 1/2$, thus $\xi_1^{2k} + \xi_2^{2k} \geqslant b^{2k} \geqslant 2^{-2k}$. Finally, if $a \leqslant b \leqslant 0$, then $|a| = -a \geqslant 1 - b \geqslant 1$ yields $\xi_1^{2k} + \xi_2^{2k} \geqslant |a|^{2k} \geqslant 1$. $\square$

Next we need to establish yet another technical lemma; recall (7) for the definition of $P_{1,m}$.

Lemma 2.3. *There exists a constant $C_1 > 0$ with the following property. Let $\varepsilon > 0$, $N \in \mathbb{N}$, and $u \in L^2$ with $\|u\|_{L^2} = 1$ be given and choose $a < b$ such that $\int_{-\infty}^a |\hat{u}(\xi)|^2\,d\xi = \varepsilon/2 = \int_b^\infty |\hat{u}(\xi)|^2\,d\xi$. Then*

$$\|T_m(\cdot)u\|_{L^4_{tx}([0,1]\times\mathbb{R})} \leqslant \left[(-P_{1,m})\left(1 - \frac{\varepsilon^2}{2}\right) + \frac{C_1 N^{q(m)}}{(b-a)^{q(m)}} + \frac{C_1 N^{2q(m)}}{(b-a)^{2q(m)}}\right]^{1/4} + C_1 N^{-1/2},$$

with $q(m) > 0$ from (13).

Proof. For a fixed u as in the assumption we divide the interval $[a, b]$ into N subintervals of equal length $(b-a)/N$. Then there must be one of the N subintervals, denoted $[a', b']$, such that $\int_{a'}^{b'} |\hat{u}(\xi)|^2\,d\xi \leqslant N^{-1}$. We introduce $u_l, u_0, u_r \in L^2$ through

$$\hat{u}_l = \mathbf{1}_{]-\infty,a'[}\,\hat{u}, \quad \hat{u}_0 = \mathbf{1}_{[a',b']}\,\hat{u}, \quad \text{and} \quad \hat{u}_r = \mathbf{1}_{]b',\infty[}\,\hat{u}.$$

It follows that $u = u_l + u_0 + u_r$ and moreover that

$$\|u_0\|_{L^2}^2 = \|\hat{u}_0\|_{L^2}^2 = \int_{a'}^{b'} |\hat{u}(\xi)|^2\,d\xi \leqslant N^{-1}.$$

Furthermore,

$$1 = \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2\,d\xi \geqslant \int_{-\infty}^{a'} |\hat{u}(\xi)|^2\,d\xi = \|u_l\|_{L^2}^2 \geqslant \int_{-\infty}^{a} |\hat{u}(\xi)|^2\,d\xi = \frac{\varepsilon}{2}.$$

In summary, taking into account the analogous bounds on $\|u_r\|_{L^2}$, we have shown that

$$\|u_0\|_{L^2} \leqslant N^{-1/2}, \quad \sqrt{\varepsilon/2} \leqslant \|u_l\|_{L^2} \leqslant 1, \quad \text{and} \quad \sqrt{\varepsilon/2} \leqslant \|u_r\|_{L^2} \leqslant 1.$$

In addition, we also have

$$\|u_l\|_{L^2}^2 + \|u_r\|_{L^2}^2 = \int_{-\infty}^{a'} |\hat{u}(\xi)|^2 \, d\xi + \int_{b'}^{\infty} |\hat{u}(\xi)|^2 \, d\xi \leqslant \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 \, d\xi = 1,$$

hence

$$\|u_l\|_{L^2}^4 + \|u_r\|_{L^2}^4 \leqslant 1 - 2\|u_l\|_{L^2}^2 \|u_r\|_{L^2}^2 \leqslant 1 - \frac{\varepsilon^2}{2}.$$

Since the supports of $\hat{u}_l$ and $\hat{u}_r$ have distance at least $b' - a' = (b-a)/N$, Lemma 2.1 implies

$$\int_0^1 \int_{\mathbb{R}} |T_m(t)u_l|^2 |T_m(t)u_r|^2 \, dx \, dt \leqslant \frac{CN^{2q}}{(b-a)^{2q}} \, \|u_l\|_{L^2}^2 \|u_r\|_{L^2}^2 \leqslant \frac{CN^{2q}}{(b-a)^{2q}},$$

where $q = q(m)$. On the other hand, by definition of $P_{1,m}$ we also have

$$\int_0^1 \int_{\mathbb{R}} |T_m(t)u_l|^4 \, dx \, dt \leqslant (-P_{1,m}) \|u_l\|_{L^2}^4 \leqslant C$$

and analogously

$$\int_0^1 \int_{\mathbb{R}} |T_m(t)u_r|^4 \, dx \, dt \leqslant (-P_{1,m}) \|u_r\|_{L^2}^4 \leqslant C.$$

From Hölder's inequality we thus deduce

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}} |T_m(t)u_l|^3 |T_m(t)u_r| \, dx \, dt \; &\leqslant \; \left(\int_0^1 \int_{\mathbb{R}} |T_m(t)u_l|^4 \, dx \, dt \right)^{1/2} \\ &\quad \times \left(\int_0^1 \int_{\mathbb{R}} |T_m(t)u_l|^2 |T_m(t)u_r|^2 \, dx \, dt \right)^{1/2} \\ &\leqslant \; \frac{CN^q}{(b-a)^q} \end{aligned}$$

and the same estimate is obtained if the roles of u_l and u_r are exchanged. Expanding

$$|T_m(t)(u_l+u_r)|^4 = |T_m(t)u_l|^4 + 4|T_m(t)u_l|^3|T_m(t)u_r| + 6|T_m(t)u_l|^2|T_m(t)u_r|^2 + 4|T_m(t)u_l||T_m(t)u_r|^3 + |T_m(t)u_r|^4$$

and invoking the above estimates, it follows that

$$\begin{aligned}\int_0^1\int_{\mathbb{R}} &|T_m(t)(u_l+u_r)|^4\,dx\,dt\\ &\leqslant (-P_{1,m})\Big(\|u_l\|_{L^2}^4 + \|u_r\|_{L^2}^4\Big) + \frac{CN^q}{(b-a)^q} + \frac{CN^{2q}}{(b-a)^{2q}}\\ &\leqslant (-P_{1,m})\Big(1-\frac{\varepsilon^2}{2}\Big) + \frac{CN^q}{(b-a)^q} + \frac{CN^{2q}}{(b-a)^{2q}}.\end{aligned}$$

If we finally take into account

$$\int_0^1\int_{\mathbb{R}} |T_m(t)u_0|^4\,dx\,dt \leqslant (-P_{1,m})\|u_0\|_{L^2}^4 \leqslant CN^{-2},$$

then the triangle inequality $\|T_m(\cdot)u\|_{L^4_{tx}([0,1]\times\mathbb{R})} \leqslant \|T_m(\cdot)(u_l+u_r)\|_{L^4_{tx}([0,1]\times\mathbb{R})} + \|T_m(\cdot)u_0\|_{L^4_{tx}([0,1]\times\mathbb{R})}$ completes the proof of the lemma. □

The next lemma is a useful consequence of Lemma 2.3.

Lemma 2.4. *For every $\varepsilon > 0$ there exist $\delta = \delta_\varepsilon > 0$ and $R = R_\varepsilon > 0$ with the following property. If $u \in L^2$ satisfies $\|u\|_{L^2} = 1$ and $\varphi(u) \leqslant P_{1,m} + \delta$, and if $a < b$ are such that $\int_{-\infty}^a |\hat{u}(\xi)|^2\,d\xi = \varepsilon/2 = \int_b^\infty |\hat{u}(\xi)|^2\,d\xi$, then $b - a \leqslant R$.*

Proof. Denote $C_1 > 0$ the constant from Lemma 2.3, and for given $\varepsilon > 0$ set

$$\delta = \frac{|P_{1,m}|\varepsilon^2}{8} \quad \text{and} \quad R = \max\left\{N, \left(\frac{16C_1N^{q(m)}}{|P_{1,m}|\varepsilon^2}\right)^{1/q(m)}\right\},$$

where $N = N_\varepsilon$ is introduced in (17) below. If $u \in L^2$ satisfies $\|u\|_{L^2} = 1$ and $\varphi(u) \leqslant P_{1,m} + \delta$, and if $a < b$ are such that $\int_{-\infty}^a |\hat{u}(\xi)|^2\,d\xi = \varepsilon/2 = \int_b^\infty |\hat{u}(\xi)|^2\,d\xi$, then $b-a>R$ cannot occur. Indeed, if $b-a>R$, then Lemma 2.3 would yield

$$\begin{aligned}\left[(-P_{1,m})\Big(1-\frac{\varepsilon^2}{8}\Big)\right]^{1/4} &= [-P_{1,m}-\delta]^{1/4}\leqslant[-\varphi(u)]^{1/4}=\|T(\cdot)u\|_{L^4_{tx}([0,1]\times\mathbb{R})}\\ &\leqslant \left[(-P_{1,m})\Big(1-\frac{\varepsilon^2}{2}\Big)+\frac{C_1N^{q(m)}}{R^{q(m)}}+\frac{C_1N^{2q(m)}}{R^{2q(m)}}\right]^{1/4}\\ &\quad +C_1N^{-1/2}\end{aligned}$$

for every $N\in\mathbb{N}$. If we select $N=N_\varepsilon\in\mathbb{N}$ such that

$$C_1N^{-1/2}\leqslant\left[(-P_{1,m})\Big(1-\frac{\varepsilon^2}{8}\Big)\right]^{1/4}-\left[(-P_{1,m})\Big(1-\frac{\varepsilon^2}{4}\Big)\right]^{1/4}, \tag{17}$$

then we obtain by definition of R the contradiction

$$(-P_{1,m})\frac{\varepsilon^2}{4}\leqslant\frac{C_1N^{q(m)}}{R^{q(m)}}+\frac{C_1N^{2q(m)}}{R^{2q(m)}}\leqslant\frac{2C_1N^{q(m)}}{R^{q(m)}}\leqslant(-P_{1,m})\frac{\varepsilon^2}{8}.$$

Hence we must in fact have $b-a\leqslant R$. □

After this preparation we can take the main step towards finding a minimizing sequence which is tight in the Fourier domain.

Lemma 2.5. *Let $m\geqslant 3$ and (u_j) be any minimizing sequence for $P_{1,m}$. Then there exist a subsequence (which is not relabelled) and $\xi_j\in\mathbb{R}$ for $j\in\mathbb{N}$ such that the following holds:*

(a) $\sup_{j\in\mathbb{N}}|\xi_j|<\infty$, *and*
(b) *for every $\varepsilon>0$ there is $R=R_\varepsilon>0$ and $j_\varepsilon\in\mathbb{N}$ so that*

$$\int_{|\xi-\xi_j|<R}|\hat{u}_j|^2\,d\xi\geqslant 1-\varepsilon,\quad j\geqslant j_\varepsilon\,.$$

Proof. For a fixed sequence $\varepsilon_k\searrow 0$ we choose $\delta_k=\delta_{\varepsilon_k}\searrow 0$ and $R_k=R_{\varepsilon_k}\nearrow\infty$ correspondingly by means of Lemma 2.4. Since (u_j) is a minimizing sequence for $P_{1,m}$, it follows from $\varphi(u_j)\to P_{1,m}$ that for every $k\in\mathbb{N}$ there is $j_k\in\mathbb{N}$ such that $\varphi(u_j)\leqslant P_{1,m}+\delta_k$ for $j\geqslant j_k$. Passing to a subsequence if necessary we therefore may assume $\varphi(u_j)\leqslant P_{1,m}+\delta_k$ for $j\geqslant k$.

Let us start by fixing $j=1$. We first select $a_1^{(1)}<b_1^{(1)}$ such that $\int_{-\infty}^{a_1^{(1)}}|\hat{u}_1|^2\,d\xi=\varepsilon_1/2=\int_{b_1^{(1)}}^{\infty}|\hat{u}_1|^2\,d\xi$. Since $\varphi(u_1)\leqslant P_{1,m}+\delta_1$, we obtain from Lemma 2.4 that $b_1^{(1)}-$

$a_1^{(1)} \leqslant R_1$. Denoting $\xi_1 = (a_1^{(1)} + b_1^{(1)})/2$ the center of the interval $[a_1^{(1)}, b_1^{(1)}]$, it follows that

$$\int_{|\xi-\xi_1|<R_1} |\hat{u}_1|^2 \, d\xi = \int_{\xi_1-R_1}^{\xi_1+R_1} |\hat{u}_1|^2 \, d\xi \geqslant \int_{a_1^{(1)}}^{b_1^{(1)}} |\hat{u}_1|^2 \, d\xi = 1 - \varepsilon_1.$$

The next step is to fix $j = 2$ and to consider u_2. First we choose $a_2^{(1)} < b_2^{(1)}$ with the property that $\int_{-\infty}^{a_2^{(1)}} |\hat{u}_2|^2 \, d\xi = \varepsilon_1/2 = \int_{b_2^{(1)}}^{\infty} |\hat{u}_2|^2 \, d\xi$. Due to $\varphi(u_2) \leqslant P_{1,m} + \delta_1$, Lemma 2.4 yields $b_2^{(1)} - a_2^{(1)} \leqslant R_1$. Next we select $a_2^{(2)} < a_2^{(1)}$ and $b_2^{(2)} > b_2^{(1)}$ such that $\int_{-\infty}^{a_2^{(2)}} |\hat{u}_2|^2 \, d\xi = \varepsilon_2/2 = \int_{b_2^{(2)}}^{\infty} |\hat{u}_2|^2 \, d\xi$. Then $\varphi(u_2) \leqslant P_{1,m} + \delta_2$ in conjunction with Lemma 2.4 implies $b_2^{(2)} - a_2^{(2)} \leqslant R_2$. We denote $\xi_2 = (a_2^{(1)} + b_2^{(1)})/2$ the center of the interval $[a_2^{(1)}, b_2^{(1)}]$. Then $b_2^{(1)} - a_2^{(1)} \leqslant R_1$ implies $\xi_2 + R_1 \geqslant b_2^{(1)}$ as well as $\xi_2 - R_1 \leqslant a_2^{(1)}$, whence

$$\int_{|\xi-\xi_2|<R_1} |\hat{u}_2|^2 \, d\xi = \int_{\xi_2-R_1}^{\xi_2+R_1} |\hat{u}_2|^2 \, d\xi \geqslant \int_{a_2^{(1)}}^{b_2^{(1)}} |\hat{u}_2|^2 \, d\xi = 1 - \varepsilon_1.$$

In addition, we also have $\xi_2 \geqslant a_2^{(1)} \geqslant a_2^{(2)}$, thus $\xi_2 + R_2 \geqslant a_2^{(2)} + R_2 \geqslant b_2^{(2)}$, and similarly $\xi_2 \leqslant b_2^{(1)} \leqslant b_2^{(2)}$ yields $\xi_2 - R_2 \leqslant b_2^{(2)} - R_2 \leqslant a_2^{(2)}$. Therefore

$$\int_{|\xi-\xi_2|<R_2} |\hat{u}_2|^2 \, d\xi = \int_{\xi_2-R_2}^{\xi_2+R_2} |\hat{u}_2|^2 \, d\xi \geqslant \int_{a_2^{(2)}}^{b_2^{(2)}} |\hat{u}_2|^2 \, d\xi = 1 - \varepsilon_2.$$

This procedure can be continued inductively to yield a sequence $(\xi_j) \subset \mathbb{R}$ such that

$$\int_{|\xi-\xi_j|<R_k} |\hat{u}_j|^2 \, d\xi \geqslant 1 - \varepsilon_k, \quad 1 \leqslant k \leqslant j,$$

holds. Then (b) is satisfied, since given $\varepsilon > 0$ we may choose $k_0 \in \mathbb{N}$ with $\varepsilon_{k_0} \leqslant \varepsilon$ and set $R = R_{k_0}$ and $j_\varepsilon = k_0$. Then $j \geqslant j_\varepsilon = k_0$ implies

$$\int_{|\xi-\xi_j|<R} |\hat{u}_j|^2 \, d\xi = \int_{|\xi-\xi_j|<R_{k_0}} |\hat{u}_j|^2 \, d\xi \geqslant 1 - \varepsilon_{k_0} \geqslant 1 - \varepsilon,$$

as was to be shown. Consequently, it remains to prove the boundedness of (ξ_j). To do so, we can assume that on the contrary there is a subsequence (not relabelled) such that $\xi_j \to \infty$; the case that $\xi_j \to -\infty$ along a subsequence can be handled similarly.

Now we fix $\varepsilon > 0$ and choose $R = R_\varepsilon > 0$ and $j_\varepsilon \in \mathbb{N}$ according to (b). Then we decompose

$$\hat{u}_j = \hat{v}_j + \hat{w}_j, \quad \text{with} \quad \hat{v}_j = \mathbf{1}_{[\xi_j - R,\, \xi_j + R]}\, \hat{u}_j, \quad j \geqslant j_\varepsilon.$$

Hence a Lipschitz estimate for φ_m, analogous to [9, (2.5)], in conjunction with $\|u_j\|_{L^2} = 1$ and Lemma 2.6 below yields for $j \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
|\varphi_m(u_j)| &\leqslant |\varphi_m(u_j) - \varphi_m(v_j)| + |\varphi_m(v_j)| \\
&\leqslant C\Big(\|u_j\|_{L^2}^3 + \|v_j\|_{L^2}^3\Big)\|u_j - v_j\|_{L^2} + C\xi_j^{-(m-2)/3}\|v_j\|_{L^2}^4 \\
&\leqslant C\|\hat{w}_j\|_{L^2} + C\xi_j^{-(m-2)/3} = C\left(\int_{|\xi-\xi_j|>R} |\hat{u}_j(\xi)|^2\, d\xi\right)^{1/2} + C\xi_j^{-(m-2)/3} \\
&= C\left(1 - \int_{|\xi-\xi_j|<R} |\hat{u}_j(\xi)|^2\, d\xi\right)^{1/2} + C\xi_j^{-(m-2)/3} \leqslant C\sqrt{\varepsilon} + C\xi_j^{-(m-2)/3}.
\end{aligned}$$

Taking the limit $j \to \infty$, this and the fact that (u_j) is a minimizing sequence gives $|P_{1,m}| \leqslant C\sqrt{\varepsilon}$ for all $\varepsilon > 0$, whence $P_{1,m} = 0$. However, similar to [9, Lemma 2.5] one can show that $P_{1,m} < 0$, which gives a contradiction. Hence we conclude that indeed (ξ_j) must be bounded. $\square$

We add two more technical results that have been used before.

Lemma 2.6. *Let $m \geqslant 3$ and φ_m be defined as in (8). If $u \in L^2$ is such that* $\operatorname{supp}(\hat{u}) \subset [\xi_* - R, \xi_* + R]$ *for some $\xi_* \geqslant \max\{1, 2R\} > 0$, then*

$$|\varphi_m(u)| \leqslant C\xi_*^{-(m-2)/3}\|u\|_{L^2}^4.$$

Proof. From (10) we recall $(T_m(t)u)(x) = \int_{\mathbb{R}} e^{i(x\xi - t\xi^m)}\hat{u}(\xi)\, d\xi$. By integrating out $\int_0^1 dt \int_{\mathbb{R}} dx$, it thus follows that

$$\begin{aligned}
|\varphi_m(u)| = \int_0^1 \int_{\mathbb{R}} |T_m(t)u|^4\, dx\, dt \;=\; &\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} d\xi_1 \ldots d\xi_4\, \hat{u}(\xi_1)\overline{\hat{u}(\xi_2)}\hat{u}(\xi_3)\overline{\hat{u}(\xi_4)} \\
&\times \delta_0(\xi_1 - \xi_2 + \xi_3 - \xi_4)\, \frac{(-i)}{\alpha}(1 - e^{-i\alpha}),
\end{aligned}$$

where

$$\alpha = \alpha(\xi_1, \ldots, \xi_4) = \xi_1^m - \xi_2^m + \xi_3^m - \xi_4^m.$$

Therefore we obtain

$$
\begin{aligned}
|\varphi_m(u)| \; = \; & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\, d\xi_2\, d\xi_3\, \hat{u}(\xi_1)\overline{\hat{u}(\xi_2)}\hat{u}(\xi_3)\overline{\hat{u}(\xi_1-\xi_2+\xi_3)} \\
& \times\frac{(-i)}{\beta}(1-e^{-i\beta}) \\
\leqslant \; & C\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\, d\xi_2\, d\xi_3\, |\hat{u}(\xi_1)||\hat{u}(\xi_2)||\hat{u}(\xi_3)||\hat{u}(\xi_1-\xi_2+\xi_3)| \\
& \times\frac{1}{1+|\beta|}\,, \qquad (18)
\end{aligned}
$$

with

$$
\beta=\beta(\xi_1,\xi_2,\xi_3)=\xi_1^m-\xi_2^m+\xi_3^m-(\xi_1-\xi_2+\xi_3)^m
$$

and we used that $|\frac{1}{\beta}(1-e^{i\beta})|\leqslant C(1+|\beta|)^{-1}$.

Case 1: m is even. We fix $\delta\in]0,1]$ and perform an argument like in [9, Lemma 2.10]. (i) On the set where $|\xi_1-\xi_2|\leqslant\delta$ we get from Young's inequality, cf. [6, Corollary 4.5.2], and with $g(-\cdot)(\xi):=g(-\xi)$

$$
\begin{aligned}
& \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\, d\xi_2\, d\xi_3\, \mathbf{1}_{\{|\xi_1-\xi_2|\leqslant\delta\}}\, |\hat{u}(\xi_1)||\hat{u}(\xi_2)||\hat{u}(\xi_3)||\hat{u}(\xi_1-\xi_2+\xi_3)|\,\frac{1}{1+|\beta|} \\
& \leqslant \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\, d\eta\, d\xi_3\, \mathbf{1}_{\{|\eta|\leqslant\delta\}}\, |\hat{u}(\xi_1)||\hat{u}(\xi_1-\eta)||\hat{u}(\xi_3)||\hat{u}(\eta+\xi_3)| \\
& \leqslant C\left\| |\hat{u}| * |\hat{u}(-\cdot)| \right\|_{L^\infty}^2 \delta \leqslant C\|u\|_{L^2}^4\delta. \qquad (19)
\end{aligned}
$$

(ii) On the set where $|\xi_2-\xi_3|\leqslant\delta$, we obtain in the same manner

$$
\begin{aligned}
& \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\, d\xi_2\, d\xi_3\, \mathbf{1}_{\{|\xi_2-\xi_3|\leqslant\delta\}}\, |\hat{u}(\xi_1)||\hat{u}(\xi_2)||\hat{u}(\xi_3)||\hat{u}(\xi_1-\xi_2+\xi_3)|\,\frac{1}{1+|\beta|} \\
& \leqslant C\|u\|_{L^2}^4\delta. \qquad (20)
\end{aligned}
$$

(iii) Now we consider the case that $|\xi_1-\xi_2|\geqslant\delta$ and $|\xi_2-\xi_3|\geqslant\delta$. Due to (18) we can always restrict our attention to $\xi_1,\xi_2,\xi_3\in\operatorname{supp}(\hat{u})$, whence $\xi_1,\xi_2,\xi_3\geqslant\xi_*-R\geqslant\xi_*/2$ by assumption. Accordingly, by Lemma 2.7(a) below we can estimate for an appropriate $\eta_0 > 0$,

$$\begin{aligned} 1+|\beta(\xi_1,\xi_2,\xi_3)| &\geqslant |\xi_1-\xi_2||\xi_2-\xi_3||\beta_{m-2}(\xi_1,\xi_2,\xi_3)| \\ &\geqslant \eta_0|\xi_1-\xi_2||\xi_2-\xi_3|\Big(|\xi_1|^{m-2}+|\xi_2|^{m-2}+|\xi_3|^{m-2}\Big) \\ &\geqslant 3\,2^{-(m-2)}\eta_0\,\xi_*^{m-2}\,|\xi_1-\xi_2||\xi_2-\xi_3| \\ &\geqslant (3/2)\delta^2\,2^{-(m-2)}\eta_0\,\xi_*^{m-2}\,(1+|\xi_1-\xi_2||\xi_2-\xi_3|). \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\,d\xi_2\,d\xi_3\,\mathbf{1}_{\{|\xi_1-\xi_2|\geqslant\delta,\,|\xi_2-\xi_3|\geqslant\delta\}}\,|\hat{u}(\xi_1)||\hat{u}(\xi_2)||\hat{u}(\xi_3)||\hat{u}(\xi_1-\xi_2+\xi_3)| \\ &\quad\times\frac{1}{1+|\beta|} \\ &\leqslant C\delta^{-2}\xi_*^{-(m-2)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\,d\xi_2\,d\xi_3\,|\hat{u}(\xi_1)||\hat{u}(\xi_2)||\hat{u}(\xi_3)||\hat{u}(\xi_1-\xi_2+\xi_3)| \\ &\quad\times\frac{1}{1+|\xi_1-\xi_2||\xi_2-\xi_3|} \\ &\leqslant C\delta^{-2}\xi_*^{-(m-2)}\|u\|_{L^2}^4, \end{aligned} \tag{21}$$

where for the last estimate one can for instance use the bound obtained from [9, (2.23)], with $\delta = 1$ and $A = 2$ there, also noting that $\hat{\Gamma}(A)\leqslant\|u\|_{L^2}^2$ for every $A > 0$ and moreover that now $\Phi \sim \hat{u}$ gives $\|\Phi\|_{L^2}\leqslant\|u\|_{L^2}$ rather than $\|\Phi\|_{L^2}\leqslant C\|u\|_{L^2}^3$, as we had in [9]. By (18), and summarizing (19)–(21), we see that

$$|\varphi_m(u)|\leqslant C\Big(\delta+\delta^{-2}\xi_*^{-(m-2)}\Big)\|u\|_{L^2}^4\leqslant C\xi_*^{-(m-2)/3}\|u\|_{L^2}^4,$$

where we have chosen the optimal $\delta = \xi_*^{-(m-2)/3}\leqslant 1$.

Case 2: m is odd. In principle we follow the same lines as before. Now we use Lemma 2.7(b) below to obtain

$$\begin{aligned} 1+|\beta(\xi_1,\xi_2,\xi_3)| &\geqslant |\xi_1-\xi_2||\xi_2-\xi_3||\xi_1+\xi_3||\beta_{m-3}(\xi_1,\xi_2,\xi_3)| \\ &\geqslant \eta_1|\xi_1-\xi_2||\xi_2-\xi_3||\xi_1+\xi_3|(|\xi_1|^{m-3}+|\xi_2|^{m-3}+|\xi_3|^{m-3}). \end{aligned}$$

Thus if $\xi_1, \xi_2, \xi_3 \geqslant \xi_* - R \geqslant \xi_*/2$, then $|\xi_1+\xi_3| = \xi_1+\xi_3 \geqslant \xi_*$, and $|\xi_1-\xi_2|, |\xi_2-\xi_3| \geqslant \delta$ yields

$$\begin{aligned} 1 + |\beta(\xi_1, \xi_2, \xi_3)| &\geqslant 3\, 2^{-(m-3)}\eta_1\, \xi_*^{m-2}\, |\xi_1 - \xi_2||\xi_2 - \xi_3| \\ &\geqslant (3/2)\delta^2\, 2^{-(m-3)}\eta_1\, \xi_*^{m-2}\, (1 + |\xi_1 - \xi_2||\xi_2 - \xi_3|). \end{aligned}$$

Hence the preceding argument can be applied once more. □

Lemma 2.7. *Let $m \in \mathbb{N}$ and*

$$\beta(x, y, z) = x^m - y^m + z^m - (x - y + z)^m, \quad x, y, z \in \mathbb{R}.$$

(a) *If m is even, then we can write $\beta(x, y, z) = (x - y)(y - z)\beta_{m-2}(x, y, z)$ with a polynomial β_{m-2} of degree $m-2$ such that $|\beta_{m-2}(x, y, z)| \geqslant \eta_0(|x|^{m-2} + |y|^{m-2} + |z|^{m-2})$ for some $\eta_0 > 0$ and all $x, y, z \in \mathbb{R}$.*

(b) *If $m \geqslant 3$ is odd, then we have $\beta(x, y, z) = (x - y)(y - z)(x + z)\beta_{m-3}(x, y, z)$, where β_{m-3} is a polynomial of degree $m-3$ so that $|\beta_{m-3}(x, y, z)| \geqslant \eta_1(|x|^{m-3} + |y|^{m-3} + |z|^{m-3})$ holds for some $\eta_1 > 0$ and all $x, y, z \in \mathbb{R}$.*

Proof. (a) We can assume that $m \geqslant 4$. First we show that $\beta(x, y, z) = 0$ implies $x = y$ or $y = z$. For this purpose we fix $y_0 \neq z_0$ and consider the function $f(x) = \beta(x, y_0, z_0)$. Then $f(y_0) = 0$ and moreover $f'(x) = mx^{m-1} - m(x - y_0 + z_0)^{m-1}$ for $x \in \mathbb{R}$. Since $(m-1)$ is even, $u \mapsto u^{m-1}$ is one-to-one on $\mathbb{R}$. Hence it follows that $f'(x) \neq 0$ for $x \in \mathbb{R}$, i.e., f is either strictly increasing of strictly decreasing. In both cases we obtain $f(x) \neq 0$ for $x \neq y_0$ as claimed, and this leads to $\beta(x, y, z) = (x-y)^k(y-z)^l B(x, y, z)$ for some maximal $k, l \in \mathbb{N}$ and a polynomial B of degree $(m - k - l)$. Differentiating both sides w.r.t. to x yields $mx^{m-1} - m(x-y+z)^{m-1} = (x-y)^{k-1}(y-z)^l[(x-y)\partial_x B + kB]$ for all $x, y, z \in \mathbb{R}$. Thus if $k \geqslant 2$, then $x = y$ enforces $mx^{m-1} - mz^{m-1} = 0$ for all $x, z \in \mathbb{R}$, which is impossible. It follows that $k = 1$, and similarly $l = 1$, so that we obtain $\beta(x, y, z) = (x - y)(y - z)\beta_{m-2}(x, y, z)$, where β_{m-2} is a polynomial of degree $m-2$. Next we claim that

$$\beta_{m-2}(x_0, y_0, z_0) = 0 \quad \Longrightarrow \quad x_0 = y_0 = z_0 = 0. \tag{22}$$

Indeed, if $\beta_{m-2}(x_0, y_0, z_0) = 0$, then also $\beta(x_0, y_0, z_0) = 0$, and consequently $x_0 = y_0$ or $y_0 = z_0$. Assuming $y_0 \neq z_0$ we can further factor $\beta_{m-2}(x, y_0, z_0) = (x - x_0)\tilde{\beta}_{m-2}(x, y_0, z_0)$ for $x \in \mathbb{R}$, so that $\beta(x, y_0, z_0) = (x - y_0)(y_0 - z_0)\beta_{m-2}(x, y_0, z_0) = (x - y_0)^2(y_0 - z_0)\tilde{\beta}_{m-2}(x, y_0, z_0)$ due to $x_0 = y_0$. Differentiating the original form of β w.r.t. to x we see that $m(x^{m-1} - (x - y_0 + z_0)^{m-1}) = (x - y_0)(y_0 - z_0)[(x - y_0)\partial_x\tilde{\beta}_{m-2} + 2\tilde{\beta}_{m-2}]$ for $x \in \mathbb{R}$, which at $x = x_0 = y_0$ yields $m(x_0^{m-1} - z_0^{m-1}) = 0$. Hence the contradiction $y_0 = x_0 = z_0$ is found. Therefore we have seen that in fact $\beta_{m-2}(x_0, y_0, z_0) = 0$ implies $x_0 = y_0 = z_0$. Differentiating $\beta(x, y, z) = (x - y)(y - z)\beta_{m-2}(x, y, z)$ w.r. to x and y, we get $m(m-1)(x - y + z)^{m-2} = (x - y)(y - z)\partial^2_{xy}\beta_{m-2} + (x - 2y + z)\partial_x\beta_{m-2} + (y - z)\partial_y\beta_{m-2} + \beta_{m-2}$. At $x_0 = y_0 = z_0$, this gives $m(m-1)z_0^{m-2} = 0$, i.e., (22) holds. Thus we must have the estimate $|\beta_{m-2}(x, y, z)| \geqslant \eta_0(|x|^{m-2} + |y|^{m-2} + |z|^{m-2})$ for some constant $\eta_0 > 0$ and all $x, y, z \in \mathbb{R}$. Otherwise there would exist sequences $(x_j), (y_j), (z_j) \subset \mathbb{R}$ and $\eta_j \to 0^+$ such that $|\beta_{m-2}(x_j, y_j, z_j)| < \eta_j(|x_j|^{m-2} + |y_j|^{m-2} + |z_j|^{m-2})$ for all $j \in \mathbb{N}$. If we assume w.l.o.g that $0 < |z_j| = \max\{|x_j|, |y_j|, |z_j|\}$ and define $\tilde{x}_j = x_j/|z_j|$, $\tilde{y}_j = y_j/|z_j|$, and $\tilde{z}_j = z_j/|z_j|$, then $|\tilde{x}_j|, |\tilde{y}_j| \leqslant 1 = |\tilde{z}_j|$, so that we can suppose that $\tilde{x}_j \to x_0$, $\tilde{y}_j \to y_0$, and $\tilde{z}_j \to z_0$ as $j \to \infty$, where $|z_0| = 1$. But $\beta(x, y, z) = (x - y)(y - z)\beta_{m-2}(x, y, z)$ shows that β_{m-2} is homogeneous of degree $m - 2$, thus as $j \to \infty$

$$
\begin{aligned}
|\beta_{m-2}(x_0, y_0, z_0)| &\leftarrow |\beta_{m-2}(\tilde{x}_j, \tilde{y}_j, \tilde{z}_j)| = |z_j|^{-(m-2)}|\beta_{m-2}(x_j, y_j, z_j)| \\
&< |z_j|^{-(m-2)}\eta_j(|x_j|^{m-2} + |y_j|^{m-2} + |z_j|^{m-2}) \\
&\leqslant 3\eta_j \to 0, \quad j \to \infty.
\end{aligned}
$$

We hence obtain $\beta_{m-2}(x_0, y_0, z_0) = 0$, which however contradicts (22) in view of $|z_0| = 1$. (b) The proof of (b) can be carried out along similar lines as in (a), so we do not expand the details. $\square$

Finally, we are in the position to show that any minimizing sequence is (up to a subsequence) tight in Fourier space.

Corollary 2.8. *Let $m \geqslant 3$ and (u_j) be any minimizing sequence for $P_{1,m}$. Then there exists a subsequence (which is not relabelled) such that the following holds: For every $\varepsilon > 0$ there is $R = R_\varepsilon > 0$ and $j_\varepsilon \in \mathbb{N}$ so that*

$$
\int_{-R}^{R} |\hat{u}_j|^2 \, d\xi \geqslant 1 - \varepsilon, \quad j \geqslant j_\varepsilon . \tag{23}
$$

Proof. Let the subsequence of (u_j) be chosen as in Lemma 2.5, and let $R_1 = \sup_{j\in\mathbb{N}} |\xi_j|$. If $\varepsilon > 0$ is given, then we set $R_\varepsilon = R_1 + \tilde{R}_\varepsilon > 0$ and $j_\varepsilon = \tilde{j}_\varepsilon \in \mathbb{N}$, where $\tilde{R}_\varepsilon$ and $\tilde{j}_\varepsilon$ are selected corresponding to ε by means of Lemma 2.5. Then $[\xi_j - \tilde{R}_\varepsilon, \xi_j + \tilde{R}_\varepsilon] \subset [-R_\varepsilon, R_\varepsilon]$ implies $\int_{-R_\varepsilon}^{R_\varepsilon} |\hat{u}_j|^2 \, d\xi \geqslant \int_{|\xi - \xi_j| < \tilde{R}_\varepsilon} |\hat{u}_j|^2 \, d\xi \geqslant 1 - \varepsilon$ for $j \geqslant j_\varepsilon$. $\square$

2.2. Tightness in physical space and convergence

In the previous section, we have shown that any minimizing sequence possesses a subsequence which is tight in Fourier space. Now, we will prove that there is yet another subsequence which (up to translation) will be tight in x-space, leading to the strong convergence (in L^2) to a minimizer. The proofs in this section are rather similar to the ones in [9], and therefore we provide details only when necessary.

We first prove one estimate which will be used to rule out the alternatives ‘vanishing’ and ‘splitting’ in the concentration compactness lemma. Since this part of the argument does not rely on the pure higher-order dispersion form, we will more generally consider $T(t)$ defined via (5), instead of $T_m(t)$ as obtained from (6).

Lemma 2.9. *Let $T(t)$ be the solution operator associated to* (5). *If $u \in H^{M-1} = H^{M-1}(\mathbb{R};\mathbb{C})$, $A > 0$, and $t \in \mathbb{R}$, then*

$$\int_{-A}^{A} |T(t)u|^2\,dx \leqslant \int_{-2A}^{2A} |u|^2\,dx + CA^{-1}|t|\,\|u\|_{H^{M-1}}^2.$$

Proof. Let $u(t,x) = (T(t)u)(x)$. From Eq. (5) we obtain

$$\partial_t(|u|^2) = 2\,\mathrm{Re}(\bar{u}u_t) = 2\,\mathrm{Im}\left(\bar{u}\sum_{m=2}^{M} b_m(-i\partial_x)^m u\right).$$

Thus if we choose a function $\zeta \in C_0^\infty(\mathbb{R})$ with values in $[0,1]$ such that $\zeta(x) = 1$ for $|x| \leqslant A$, $\zeta(x) = 0$ for $|x| \geqslant 2A$, and $\|\zeta'\|_{L^\infty(\mathbb{R})} \leqslant CA^{-1}$, then is follows with $I(t) = \int_{\mathbb{R}} \zeta(x)|u(t,x)|^2\,dx$ that

$$\dot{I}(t) = 2\sum_{m=2}^{M} b_m\,\mathrm{Im}\left((-i)^m \int_{\mathbb{R}} \zeta\bar{u}\,\partial_x^m u\,dx\right).$$

Now

$$\int_{\mathbb{R}} \zeta\bar{u}\,\partial_x^m u\,dx = -\int_{\mathbb{R}} [\zeta'\bar{u} + \zeta(\partial_x\bar{u})]\,\partial_x^{m-1}u\,dx =: J_1(t) - \int_{\mathbb{R}} \zeta(\partial_x\bar{u})\,\partial_x^{m-1}u\,dx,$$

where $|J_1(t)| \leqslant CA^{-1}\|u(t)\|_{L^2}\|u(t)\|_{H^{M-1}} \leqslant CA^{-1}\|u\|_{H^{M-1}}^2$; for the latter estimate, note that $\widehat{u(t)}(\xi) = e^{-it(\sum_{m=2}^{M} b_m\xi^m)}\hat{u}(\xi)$, whence $\|u(t)\|_{H^s} = \|u\|_{H^s}$ for $s \in \mathbb{R}$. Then we may continue

$$\begin{aligned}\int_{\mathbb{R}} \zeta\bar{u}\,\partial_x^m u\,dx \;=\; & J_1(t) + \int_{\mathbb{R}} [\zeta'(\partial_x\bar{u}) + \zeta(\partial_x^2\bar{u})]\,\partial_x^{m-2}u\,dx =: J_1(t) + J_2(t) \\ & + \int_{\mathbb{R}} \zeta(\partial_x^2\bar{u})\,\partial_x^{m-2}u\,dx,\end{aligned}$$

where again $|J_2(t)| \leqslant CA^{-1}\|u\|^2_{H^{M-1}}$. Thus the repeated application of this procedure finally yields

$$\int_{\mathbb{R}} \zeta \bar{u}\, \partial_x^m u\, dx = J(t) + (-1)^m \int_{\mathbb{R}} \zeta (\partial_x^m \bar{u}) u\, dx,$$

with $|J(t)| \leqslant CA^{-1}\|u\|^2_{H^{M-1}}$. Therefore

$$\dot{I}(t) = 2 \sum_{m=2}^{M} b_m \operatorname{Im}\Big((-i)^m \int_{\mathbb{R}} \zeta \bar{u}\, \partial_x^m u\, dx \Big) = \sum_{m=2}^{M} b_m \operatorname{Im}\Big((-i)^m J(t) \Big)$$

leads to $|\dot{I}(t)| \leqslant CA^{-1}\|u\|^2_{H^{M-1}}$. Hence for $t \geqslant 0$,

$$\begin{aligned} \int_{-A}^{A} |T(t)u|^2 dx \; &\leqslant \; \int_{\mathbb{R}} \zeta |u(t)|^2\, dx = I(t) = I(0) + \int_0^t \dot{I}(s)\, ds \\ &\leqslant \; \int_{\mathbb{R}} \zeta |u|^2\, dx + CA^{-1}|t|\, \|u\|^2_{H^{M-1}}, \end{aligned}$$

which implies the required estimate. $\square$

Following the lines of [9, Lemma 2.7], one then establishes the next estimate.

Lemma 2.10. *For $u \in H^{M-1}$, $t \in [0,1]$, and $A \geqslant 1$ we have*

$$\int_{\mathbb{R}} |T(t)u|^4\, dx \leqslant C \bigg(\sup_{x_0 \in \mathbb{R}} \int_{x_0-2A}^{x_0+2A} |u|^2\, dx + A^{-1}\|u\|^2_{H^{M-1}} \bigg) \|u\|^2_{H^1}.$$

Now, we are ready complete the proof of Theorem 1.1. Our argument varies only slightly from the original one in [9]. From Corollary 2.8 we already know that by passing to a subsequence of any minimizing sequence (u_j), we may assume that $(\hat{u}_j)$ is tight, in the sense of (23). Then the concentration compactness lemma is applied to (u_j), see [10] or [9, Lemma 3.1] for the form which is to be used here. This leads to three alternatives for (a further subsequence of) the sequence (u_j), namely ‘tightness’, ‘vanishing’, or ‘splitting’. In the first case one can follow the reasoning in [9, Section 4.1.1] to prove that (u_j) has a strong limit in L^2, which then yields the desired minimizer for $P_{1,m}$. This argument only relies on the shift invariance $\varphi_m(u(\cdot + x_0)) = \varphi_m(u)$, which holds here, since $(T(t)u(\cdot + x_0))(x) = (T(t)u)(x + x_0)$ is a consequence of the fact that both sides have Fourier transform $e^{ix_0\xi - it(\sum_{m=2}^{M} b_m \xi^m)}\hat{u}(\xi)$. Finally, to rule out ‘vanishing’ one can just copy the argument given in [9, Section 4.1.2] using Lemma 2.10, and that ‘splitting’ is impossible may be verified as in [9, Section 4.1.3].

3. Proof of Theorem 1.2

Given the similarity of Theorem 1.2 to Theorem 1.1, we do only point out which modifications are necessary to carry through the argument elaborated in Section 2. Lemma 2.1 has to be replaced by the following:

Lemma 3.1. *There exists a constant* $C > 0$ *such that*

$$\|(T(\cdot)u)(T(\cdot)v)\|_{L^2_{tx}([0,1]\times\mathbb{R})} \leqslant C\,\mathrm{dist}(I,J)^{-1/6}\,\|u\|_{L^2}\|v\|_{L^2}$$

for all functions $u, v \in L^2$ *such that* $\hat{u}$ *and* $\hat{v}$ *are supported in disjoint intervals* $I \subset \mathbb{R}$ *and* $J \subset \mathbb{R}$, *respectively, which are at positive distance.*

Proof. The relation

$$u(t,x) = (T(t)u)(x) = \int_{\mathbb{R}} e^{ix\xi} e^{-it(b_2\xi^2+b_3\xi^3)}\hat{u}(\xi)\,d\xi \tag{24}$$

yields, in the notation of Lemma 2.1,

$$\Phi(\tau,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{u}(\xi_1)\hat{v}(\xi_2)\delta_0(\tau+\sigma(\xi_1)+\sigma(\xi_2))\delta_0(\xi-\xi_1-\xi_2)d\xi_1\,d\xi_2,$$

whence

$$\begin{aligned}&\|(T(\cdot)u)(T(\cdot)v)\|^2_{L^2_{tx}([0,1]\times\mathbb{R})}\\ &\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{u}(\xi_1)\hat{v}(\xi_2)G(-\sigma(\xi_1)-\sigma(\xi_2),\xi_1+\xi_2)\,d\xi_1\,d\xi_2,\end{aligned}$$

where $\sigma(\xi) = b_2\xi^2 + b_3\xi^3$. Then we proceed as in Lemma 2.1 in the Case 2 (m odd) and insert the factor $|\sigma'(\xi_1)-\sigma'(\xi_2)|^{-1/3+1/3}$ into the integral. To estimate the second resulting term

$$R_2 = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma'(\xi_1)-\sigma'(\xi_2)||G(-\sigma(\xi_1)-\sigma(\xi_2),\xi_1+\xi_2)|^3\,d\xi_1\,d\xi_2\right)^{1/3},$$

we introduce the transformation $(\eta_1,\eta_2) = (-\sigma(\xi_1)-\sigma(\xi_2),\xi_1+\xi_2)$, which leads to $R_2 \leqslant C\|G\|_{L^3_{\tau\xi}} \leqslant C\|u\|_{L^2}\|v\|_{L^2}$ as before. For the first resulting term

$$R_1 = \left(\int_I\int_J \frac{|\hat{u}(\xi_1)|^{3/2}|\hat{v}(\xi_2)|^{3/2}}{|\sigma'(\xi_1)-\sigma'(\xi_2)|^{1/2}}\,d\xi_1\,d\xi_2\right)^{2/3},$$

we observe that $|\sigma'(\xi_1)-\sigma'(\xi_2)| = |3b_3(\xi_1^2-\xi_2^2)+2b_2(\xi_1-\xi_2)| = |\xi_1-\xi_2||3b_3(\xi_1+\xi_2)+2b_2| \geqslant C(b-a)|\xi_1+\xi_2+\gamma|$ for $\xi_1 \in I$ and $\xi_2 \in J$, with $\gamma = 2b_2/(3b_3)$. Then in the subsequent application of the Hardy–Littlewood–Sobolev inequality the constant γ can be absorbed through e.g. the transformation $(\eta_1,\eta_2) = (\xi_1, \xi_2+\gamma)$. Hence it is found that $\|(T(\cdot)u)(T(\cdot)v)\|^2_{L^2_{tx}([0,1]\times\mathbb{R})} \leqslant CR_1R_2 \leqslant C\,\mathrm{dist}(I,J)^{-1/3}\,\|u\|^2_{L^2}\|v\|^2_{L^2}$, as before. $\square$

The only other place in Section 2.1 where the particular form $\sigma(\xi) = \xi^m$ of the dispersion function in the pure higher-order dispersion case was used is Lemma 2.6. Accordingly, we have to derive an appropriate modification for the mixed case considered here, where we have $\sigma(\xi) = b_2\xi^2 + b_3\xi^3$.

Lemma 3.2. *Let φ be given by* (9). *If $u \in L^2$ is such that* $\mathrm{supp}(\hat{u}) \subset [\xi_*-R, \xi_*+R]$ *for some* $\xi_* \geqslant \max\{1, 2R, 2|b_2|/|b_3|\} > 0$, *then*

$$|\varphi(u)| \leqslant C\xi_*^{-1/3}\|u\|^4_{L^2}.$$

Proof. From (24) one deduces in analogy to (18),

$$|\varphi(u)| \leqslant C\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_1\,d\xi_2\,d\xi_3\,|\hat{u}(\xi_1)||\hat{u}(\xi_2)||\hat{u}(\xi_3)||\hat{u}(\xi_1-\xi_2+\xi_3)|\,\frac{1}{1+|\beta|}\,,$$

where

$$\beta = \beta(\xi_1,\xi_2,\xi_3) = \sigma(\xi_1)-\sigma(\xi_2)+\sigma(\xi_3)-\sigma(\xi_1-\xi_2+\xi_3).$$

With $\sigma(\xi) = b_2\xi^2 + b_3\xi^3$, this is evaluated as

$$\beta = (\xi_1-\xi_2)(\xi_2-\xi_3)\Big(2b_2+3b_3(\xi_1+\xi_3)\Big)$$

and if $|\xi_1-\xi_2|, |\xi_2-\xi_3| \geqslant \delta$ and $\xi_1,\xi_2,\xi_3 \geqslant \xi_*-R \geqslant \xi_*/2$ as well as $\xi_* \geqslant 2|b_2|/|b_3|$, then for $\delta \in]0,1]$,

$$\begin{aligned}
1+|\beta| &\geqslant |\xi_1-\xi_2||\xi_2-\xi_3|\Big(3|b_3|(\xi_1+\xi_3)-2|b_2|\Big)\\
&\geqslant |\xi_1-\xi_2||\xi_2-\xi_3|\Big(3|b_3|\xi_*-2|b_2|\Big)\\
&\geqslant 2|b_3|\,|\xi_1-\xi_2||\xi_2-\xi_3|\,\xi_* \geqslant \delta^2|b_3|\,(1+|\xi_1-\xi_2||\xi_2-\xi_3|)\,\xi_*.
\end{aligned}$$

Therefore, it is clear that the argument from Lemma 2.6 can be applied to obtain the desired estimate. □

Since we have seen that the necessary modifications compared to Section 2.1 are possible, it follows as in Corollary 2.8 that any minimizing sequence (u_j) for P_1 has a subsequence (which is not relabelled) such that $(\hat{u}_j)$ is tight, in the sense of (23). Next we observe that concerning the application of the concentration compactness lemma to (u_j) in Section 2.2, we already established Lemmas 2.9 and 2.10 for the general mixed dispersion case, i.e., for $T(t)$ defined via (5). Thus these results in particular are valid in the mixed third-order case which is considered here. Hence one can follow the reasoning which is outlined in Section 2.2 and elaborated in [9] to complete the proof of Theorem 1.2. □

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